ON THE NON-EXISTENCE OF SIMPLE CONGRUENCES FOR QUOTIENTS OF EISENSTEIN SERIES

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Abstract. A recent article of Berndt and Yee found congruences modulo 3^k for certain ratios of Eisenstein series. For all but one of these, we show there are no simple congruences $a(\ell n+c) \equiv 0 \pmod{\ell}$ when $\ell \geq 13$ is prime. This follows from a more general theorem on the non-existence of congruences in $E_2^r E_4^s E_6^t$ where $r \geq 0$ and $s, t \in \mathbb{Z}$.

1. Introduction

Define $p(n)$ to be the number of ways of writing n as a sum of non-increasing positive integers. Ramanujan famously established the congruences

$$p(5n+4) \equiv 0 \pmod{5}$$
$$p(7n+5) \equiv 0 \pmod{7}$$
$$p(11n+6) \equiv 0 \pmod{11}$$

and noted that there does not appear to be any other prime for which the partition function has equally simple congruences. Ahlgren and Boylan [1] build on the work of Kiming and Olsson [5] to prove that there truly are no other such primes. For large enough primes ℓ, Sinick [7] and the author [3] prove the non-existence of simple congruences

$$a(\ell n+c) \equiv 0 \pmod{\ell}$$

for wide classes of functions $a(n)$ related to the coefficients of modular forms. However, all of the modular forms studied in [1], [7] and [3] are non-vanishing on the upper half plane. Here we prove the non-existence of simple congruences (when ℓ is large enough) for ratios of Eisenstein series.

Let $\sigma_m(n) := \sum_{d|n} d^m$ and define the Bernoulli numbers B_k by $\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$. For even $k \geq 2$, set

$$E_k(\tau) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Note that $E_2 \equiv E_4 \equiv E_6 \equiv 1$ modulo 2 and 3. Berndt and Yee [2] prove congruences for the quotients of Eisenstein series in Table 1 below, where $F(q) := \sum a(n)q^n$. An obviously necessary requirement for the congruences in the $n \equiv 2 \pmod{3}$ column of Table 1 is that there are simple congruences of the form $a(3n+2) \equiv 0 \pmod{3}$. All but the first form in Table 1 are covered by the following theorem.

Theorem 1.1. *Let $r \geq 0$ and $s, t \in \mathbb{Z}$. If $E_2^r E_4^s E_6^t = \sum a(n)q^n$ has a simple congruence $a(\ell n+c) \equiv 0 \pmod{\ell}$ for the prime ℓ, then either $\ell \leq 2r + 8|s| + 12|t| + 21$ or $r = s = t = 0$.*

This theorem gives an explicit upper bound on primes ℓ for which there can be congruences of the form $a(\ell n+c) \equiv 0 \pmod{\ell^k}$ as in the middle column of Table 1.

Table 1. Congruences of Berndt and Yee [2]

$F(q)$	$n \equiv 2 \pmod 3$	$n \equiv 4 \pmod 8$
$1/E_2$	$a(n) \equiv 0 \pmod{3^4}$	
$1/E_4$	$a(n) \equiv 0 \pmod{3^2}$	
$1/E_6$	$a(n) \equiv 0 \pmod{3^3}$	$a(n) \equiv 0 \pmod{7^2}$
E_2/E_4	$a(n) \equiv 0 \pmod{3^3}$	
E_2/E_6	$a(n) \equiv 0 \pmod{3^2}$	$a(n) \equiv 0 \pmod{7^2}$
E_4/E_6	$a(n) \equiv 0 \pmod{3^3}$	
E_2^2/E_6	$a(n) \equiv 0 \pmod{3^5}$	

Remark 1.2. *See Remark 4.1 for a slight improvement of Theorem 1.1 in some cases.*

Example 1.3. *The form E_6/E_4^{12} can only have simple congruences for $\ell \leq 129$. Of these, the primes $\ell = 2$ and 3 are trivial with $E_4 \equiv E_6 \equiv 1 \pmod{\ell}$. For the remaining primes, the only congruences are*

$$a(\ell n + c) \equiv 0 \pmod{17}, \quad \textit{where} \ \left(\frac{c}{17}\right) = -1.$$

Mahlburg [6] shows that for each of the forms in Table 1 except $1/E_2$, there are infinitely many primes ℓ such that for any $i \geq 1$, the set of n with $a(n) \equiv 0 \pmod{\ell^i}$ has arithmetic density 1. On the other hand, our result shows that (for large enough ℓ) every arithmetic progression modulo ℓ has at least one non-vanishing coefficient modulo ℓ.

Section 2 recalls certain definitions and tools from the theory of modular forms. Simple congruences are reinterpreted in terms of Tate cycles, which are reviewed in Section 3. Section 4 proves Theorem 1.1.

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2. Preliminaries

A modular form of weight $k \in \mathbb{Z}$ on $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which satisfies

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, and which is holomorphic at infinity. Modular forms have Fourier expansions in powers of $q = e^{2\pi i \tau}$. For any prime $\ell \geq 5$, let $\mathbb{Z}_{(\ell)} = \{\frac{a}{b} \in \mathbb{Q} : \ell \nmid b\}$. We denote the set of all weight k modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with ℓ-integral Fourier coefficients by M_k. Although E_k is a modular form of weight k whenever $k \geq 4$, E_2 is called a quasi-modular form since it satisfies the slightly different transformation rule

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6ic}{\pi}(c\tau + d).$$

Definition. *If ℓ is a prime, a Laurent series $f = \sum_{n \geq N} a(n) q^n \in \mathbb{Z}_{(\ell)}((q))$ has a simple congruence at $c \pmod{\ell}$ if $a(\ell n + c) \equiv 0 \pmod{\ell}$ for all n.*

Lemma 2.1. *Suppose that ℓ is prime and that $f = \sum a(n)q^n$ and $g = \sum b(n)q^n \in \mathbb{Z}_{(\ell)}((q))$ with $g \not\equiv 0 \pmod{\ell}$. The series f has a simple congruence at $c \pmod{\ell}$ if and only if the series fg^ℓ has a simple congruence at $c \pmod{\ell}$.*

Proof. It suffices to consider the reductions $\pmod{\ell}$ of the series

$$\left(\sum a(n)q^n\right)\left(\sum b(n)q^{\ell n}\right) \equiv \sum_n \left(\sum_m b(m)a(n-\ell m)\right) q^n \pmod{\ell}.$$

If $a(n)$ vanishes when $n \equiv c \pmod{\ell}$, then the inner sum on the right hand side will also vanish for $n \equiv c \pmod{\ell}$. The converse follows via multiplication by $\left(\sum b(n)q^n\right)^{-\ell}$ and repetition of this argument. $\square$

Our main tool is Ramanujan's Θ operator

$$\Theta := \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}.$$

For any prime ℓ and any Laurent series $f = \sum a(n)q^n \in \mathbb{Z}_{(\ell)}((q))$, by Fermat's Little Theorem

$$\Theta^\ell f = \sum a(n)n^\ell q^n \equiv \sum a(n)nq^n = \Theta f \pmod{\ell}.$$

We call the sequence $\Theta f, \ldots, \Theta^\ell f \pmod{\ell}$ the Tate cycle of f. Note that $\Theta^{\ell-1} f \equiv f \pmod{\ell}$ is equivalent to f having a simple congruence at $0 \pmod{\ell}$.

We now recall some facts about the reductions of modular forms $\pmod{\ell}$. See Swinnerton-Dyer [8] Section 3 for the details on this paragraph. There are polynomials $A(Q,R), B(Q,R) \in \mathbb{Z}_{(\ell)}[Q,R]$ such that

$$A(E_4, E_6) = E_{\ell-1},$$
$$B(E_4, E_6) = E_{\ell+1}.$$

Reduce the coefficients of these polynomials modulo ℓ to get $\tilde{A}, \tilde{B} \in \mathbb{F}_\ell[Q,R]$. Then $\tilde{A}$ has no repeated factor and is prime to $\tilde{B}$. Furthermore, the $\mathbb{F}_\ell$-algebra of reduced modular forms is naturally isomorphic to

$$\frac{\mathbb{F}_\ell[Q,R]}{\tilde{A}-1} \tag{2.1}$$

via $Q \to E_4$ and $R \to E_6$. Whenever a power series f is congruent to a modular form, define the filtration of f by

$$\omega(f) := \inf\{k : f \equiv g \in M_k \pmod{\ell}\}.$$

If $f \in M_k$, then for some $g \in M_{k+\ell+1}$, $\Theta f \equiv g \pmod{\ell}$. The next lemma also follows from [8] Section 3.

Lemma 2.2. *Let $\ell \geq 5$ be prime, $f \in M_{k_1}$, $f \not\equiv 0 \pmod{\ell}$ and $g \in M_{k_2}$.*

(1) If $f \equiv g \pmod{\ell}$ then $k_1 \equiv k_2 \pmod{\ell - 1}$,
(2) $\omega(\Theta f) \leq \omega(f) + \ell + 1$ with equality if and only if $\omega(f) \not\equiv 0 \pmod{\ell}$,
(3) If $\omega(f) \equiv 0 \pmod{\ell}$, then for some $s \geq 1$, $\omega(\Theta f) = \omega(f) + (\ell+1) - s(\ell-1)$, and
(4) $\omega(f^i) = i\omega(f)$.

The natural grading induced by (2.1) provides a key step in the following lemma which is taken from the proof of [5] Proposition 2.

Lemma 2.3. *A form $f \in M_k$ with $\Theta f \not\equiv 0 \pmod{\ell}$ has a simple congruence at $c \not\equiv 0 \pmod{\ell}$ if and only if $\Theta^{\frac{\ell+1}{2}} f \equiv -\left(\frac{c}{\ell}\right) \Theta f \pmod{\ell}$.*

Proof. Since Θ satisfies the product rule,

$$\begin{aligned}
\Theta^{\ell-1}\left(q^{-c} f\right) &\equiv \sum_{i=0}^{\ell-1} \binom{\ell-1}{i} (-c)^{\ell-1-i} q^{-c} \Theta^i f \pmod{\ell} \\
&\equiv \sum_{i=0}^{\ell-1} c^{\ell-1-i} q^{-c} \Theta^i f \pmod{\ell} \\
&\equiv c^{\ell-1} q^{-c} f + \sum_{i=1}^{\ell-1} c^{\ell-1-i} q^{-c} \Theta^i f \pmod{\ell}.
\end{aligned}$$

A simple congruence for f at $c \not\equiv 0 \pmod{\ell}$ is equivalent to a simple congruence for $q^{-c} f$ at $0 \pmod{\ell}$, which in turn is equivalent to $\Theta^{\ell-1}(q^{-c} f) \equiv q^{-c} f \pmod{\ell}$. By the computation above, this is equivalent to $0 \equiv \sum_{i=1}^{\ell-1} c^{\ell-1-i} q^{-c} \Theta^i f \pmod{\ell}$, and hence to $0 \equiv \sum_{i=1}^{\ell-1} c^{\ell-1-i} \Theta^i f \pmod{\ell}$. By Lemma 2.2 (2) and (3), for $1 \le i \le \frac{\ell-1}{2}$ we have

$$\omega(\Theta^i f) \equiv \omega(\Theta^{i+\frac{\ell-1}{2}} f) \equiv \omega(f) + 2i \pmod{\ell - 1}.$$

By Lemma 2.2 (1) and the natural grading (filtration modulo $\ell - 1$), the only way for the given sum to be zero is if for all $1 \le i \le \frac{\ell-1}{2}$ we have

$$c^{\ell-1-i} \Theta^i f + c^{\ell-1-(i+\frac{\ell-1}{2})} \Theta^{i+\frac{\ell-1}{2}} f \equiv 0 \pmod{\ell},$$

which happens if and only if

$$\Theta^{i+\frac{\ell-1}{2}} f \equiv -c^{\frac{\ell-1}{2}} \Theta^i f \equiv -\left(\frac{c}{\ell}\right) \Theta^i f \pmod{\ell},$$

which happens if and only if

$$\Theta^{\frac{\ell+1}{2}} f \equiv -\left(\frac{c}{\ell}\right) \Theta f \pmod{\ell}.$$

□

Lemma 2.4. *Let $a, b, c \ge 0$ be integers and let $\ell > 11$ be prime. Then $\omega(E_{\ell+1}^a E_4^b E_6^c) = a\ell + a + 4b + 6c$.*

Proof. Since $E_{\ell+1}^a E_4^b E_6^c \in M_{a\ell+a+4b+6c}$, it suffices to show that $\tilde{A}(Q, R)$ does not divide $\tilde{B}(Q, R)^a Q^b R^c$. However $\tilde{A}$ has no repeated factors and is prime to $\tilde{B}$ and so it suffices to show that $\tilde{A}$ does not divide QR. But QR has weight 10 and $E_{\ell-1}$ has weight $\ell - 1 > 10$ so this is impossible. □

3. The Structure of Tate Cycles

The following framework follows Jochnowitz [4]. Let $f \in M_k$ be such that $\Theta f \not\equiv 0 \pmod{\ell}$. Recall from Section 2 that the Tate cycle of f is the sequence $\Theta f, \ldots, \Theta^{\ell-1} f \pmod{\ell}$. By Lemma 2.2 (2) and (3),

$$\omega(\Theta^{i+1} f) \equiv \begin{cases} \omega(\Theta^i f) + 1 \pmod{\ell} & \text{if } \omega(\Theta^i f) \not\equiv 0 \pmod{\ell} \\ s + 1 \pmod{\ell} & \text{if } \omega(\Theta^i f) \equiv 0 \pmod{\ell}, \end{cases}$$

for some $s \geq 1$. In particular, when $\omega(\Theta^i f) \equiv 0 \pmod{\ell}$, the amount s by which the filtration decreases controls when the *next* decrease occurs. We say that $\Theta^i f$ is a high point of the Tate cycle and $\Theta^{i+1} f$ is a low point of the Tate cycle whenever $\omega(\Theta^i f) \equiv 0 \pmod{\ell}$. Elementary considerations (see, for example, [4] Section 7 or [3] Section 3) yield

Lemma 3.1. *Let $f \in M_k$ with $\Theta f \not\equiv 0 \pmod{\ell}$.*

(1) If the Tate cycle has only one low point, then the low point has filtration $2 \pmod{\ell}$.
(2) The Tate cycle has one or two low points.

Lemma 3.2. *Suppose $f \in M_k$ has a simple congruence at $c \not\equiv 0 \pmod{\ell}$, where $\ell \geq 5$ is prime, and $\Theta f \not\equiv 0 \pmod{\ell}$. Then the Tate cycle of f has two low points. Furthermore, if $\Theta^i f$ is a high point, then*

$$\omega(\Theta^{i+1} f) = \omega(\Theta^i f) + (\ell+1) - \left(\frac{\ell+1}{2}\right)(\ell-1) \equiv \frac{\ell+3}{2} \pmod{\ell}.$$

Proof. By Lemma 2.3, $\omega\left(\Theta f\right) = \omega(\Theta^{\frac{\ell+1}{2}} f)$. Hence, the filtration is not monotonically increasing between Θf and $\Theta^{\frac{\ell+1}{2}} f$, so there must be a fall in filtration somewhere in the first half of the Tate cycle. We also have $\omega(\Theta^{\frac{\ell+1}{2}} f) = \omega\left(\Theta f\right) = \omega\left(\Theta^\ell f\right)$ and so there must be a low point somewhere in the second half of the Tate cycle. By Lemma 3.1, there are exactly two low points in the Tate cycle. Lemma 2.2 (2) and (3) give

$$\omega\left(\Theta f\right) = \omega\left(\Theta^{\frac{\ell+1}{2}} f\right) = \omega\left(\Theta f\right) + \left(\frac{\ell-1}{2}\right)(\ell+1) - s(\ell-1)$$

for some $s \geq 1$. Hence $s = \frac{\ell+1}{2}$. The lemma follows. □

The proof of Theorem 1.1 uses the previous lemma to determine how far the filtration falls, and the bounds of the next lemma to show a corresponding restriction on ℓ.

Lemma 3.3. *Let $\ell \geq 5$ be prime and suppose $f \in M_k$ has a simple congruence at $c \not\equiv 0 \pmod{\ell}$. If $\omega(f) = A\ell + B$ where $1 \leq B \leq \ell - 1$, then*

$$\frac{\ell+1}{2} \leq B \leq A + \frac{\ell+3}{2}.$$

Proof. Since $B \neq 0$, $\omega(\Theta f) = (A+1)\ell + (B+1)$. From the proof of Lemma 3.2, the Tate cycle has a high point before $\Theta^{\frac{\ell+1}{2}} f$. Hence by Lemma 2.2 (2),

$$B + 1 + \frac{\ell-3}{2} \geq \ell,$$

which gives the first inequality. Also by Lemma 2.2, the high point has filtration

$$\begin{aligned} \omega(\Theta^{\ell-B} f) &= \omega(f) + (\ell-B)(\ell+1) \\ &= (A+\ell-B+1)\ell. \end{aligned}$$

Lemma 3.2 implies that the corresponding low point has filtration

$$\omega(\Theta^{\ell-B+1} f) = \left(A - B + \frac{\ell+3}{2}\right)\ell + \left(\frac{\ell+3}{2}\right).$$

The fact that $\omega(\Theta^{\ell-B+1} f) \geq 0$ implies the second inequality. □

If $\Theta f \equiv 0 \pmod{\ell}$ then the Tate cycle is trivial and above lemmas are not applicable. We dispense with this case now.

Lemma 3.4. *Let $f = E_2^r E_4^s E_6^t$ where $r \geq 0$ and $s, t \in \mathbb{Z}$. If ℓ is a prime such that $\Theta f \equiv 0 \pmod{\ell}$ then either $\ell \leq 13$ or $r \equiv s \equiv t \equiv 0 \pmod{\ell}$.*

Example 3.5. *We have $\Theta(E_4 E_6) \equiv 0 \pmod{\ell}$ for $\ell = 2, 3, 11$.*

Example 3.6. *We have $\Theta(E_2^{144} E_4^{-15} E_6^{-14}) \equiv 0 \pmod{\ell}$ for $\ell = 2, 3, 5, 7, 13$.*

Note that $\Theta f \equiv 0 \pmod{\ell}$ is equivalent to f having simple congruences at all $c \not\equiv 0 \pmod{\ell}$.

Proof of Lemma 3.4. Assume $\ell \geq 17$ and expand f as a power series to get

$$f = 1 + (-24r + 240s - 504t)q + (288r^2 - 5760rs + 12096rt \\ - 360r + 28800s^2 - 120960st - 26640s + 127008t^2 - 143640t)q^2 + \cdots .$$

If $\Theta f \equiv 0 \pmod{\ell}$, then the coefficients of q and q^2 vanish modulo ℓ. That is,

$$-24r + 240s - 504t \equiv 0 \pmod{\ell}, \tag{3.1}$$

and

$$\begin{matrix} 288r^2 - 5760rs + 12096rt - 360r + 28800s^2 \\ -120960st - 26640s + 127008t^2 - 143640t \end{matrix} \equiv 0 \pmod{\ell}. \tag{3.2}$$

Furthermore, by Lemmas 2.2(2) and 2.4 and the fact that $E_2 \equiv E_{\ell+1} \pmod{\ell}$, we have

$$\omega(E_{\ell+1}^r E_4^s E_6^t) \equiv r + 4s + 6t \equiv 0 \pmod{\ell}. \tag{3.3}$$

Solving the system of congruences given by (3.3) and (3.1) yields

$$7r \equiv -72t \pmod{\ell}, \tag{3.4}$$

$$14s \equiv 15t \pmod{\ell}. \tag{3.5}$$

Substituting (3.4) and (3.5) into 49 times (3.2) yields

$$-8255520t \equiv 0 \pmod{\ell}.$$

Since $8255520 = 2^5 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$, the lemma follows. □

4. Proof of Theorem 1.1

We begin with the trivial observation that $E_2^r E_4^s E_6^t = 1 + \cdots$ does not have a simple congruence at $0 \pmod{\ell}$. Hence, we assume that $E_2^r E_4^s E_6^t$ has a simple congruence at $c \not\equiv 0 \pmod{\ell}$, where $\ell \geq 5$. Since $E_2 \equiv E_{\ell+1} \pmod{\ell}$, $E_{\ell+1}^r E_4^s E_6^t$ has a simple congruence at $c \pmod{\ell}$. Recall that our goal is to show $\ell \leq 2r + 8|s| + 12|t| + 21$. Hence, if $\ell < |s|$ or $\ell < |t|$ then we are done. Thus we assume $\ell + s \geq 0$ and $\ell + t \geq 0$. We also assume $\ell > 11$. Lemma 3.4 allows us to take $\Theta(E_2^r E_4^s E_6^t) \not\equiv 0 \pmod{\ell}$ (otherwise we are done). By Lemma 2.1 we see that

$$E_{\ell+1}^r E_4^{\ell+s} E_6^{\ell+t} \in M_{(r+10)\ell+(r+4s+6t)}$$

has a simple congruence at $c \pmod{\ell}$. By Lemma 2.4,

$$\omega(E_{\ell+1}^r E_4^{\ell+s} E_6^{\ell+t}) = (r+10)\ell + (r + 4s + 6t). \tag{4.1}$$

We break into four cases depending on the size of $r + 4s + 6t$:

(1) If $\ell \leq |r + 4s + 6t|$ then we are done.
(2) If $0 < r + 4s + 6t < \ell$ then by Equation (4.1) and the first inequality of Lemma 3.3, $\frac{\ell+1}{2} \leq r + 4s + 6t$ and we are done.

(3) If $r+4s+6t=0$, then by Lemma 2.2
$$\omega(\Theta E_{\ell+1}^r E_4^{\ell+s} E_6^{\ell+t}) = (r+11)\ell + 1 - s'(\ell-1)$$
for some $1 \leq s'$. If $\ell > r+13$, then in order for this filtration to be non-negative, $s' \leq r+11$. Now $\omega(\Theta E_{\ell+1}^r E_4^{\ell+s} E_6^{\ell+t}) \equiv s'+1 \pmod{\ell}$. By Lemma 2.3, there must be a high point of the Tate cycle before $\Theta^{\frac{\ell+1}{2}} E_{\ell+1}^r E_4^{\ell+s} E_6^{\ell+t}$. Hence
$$s' + 1 + \frac{\ell-3}{2} \geq \ell.$$
That is, $\ell \leq 2s'-1 \leq 2r+21$ and we are done.

(4) If $-\ell < r+4s+6t < 0$, then take $B = \ell + r + 4s + 6t$ and $A = r+9$. Equation (4.1) and the second inequality of Lemma 3.3 gives
$$\ell + r + 4s + 6t \leq r + 9 + \frac{\ell+3}{2}$$
which is equivalent to $\ell \leq 21 - 8s - 12t$ and we are done.

Remark 4.1. *Combining these four cases and recalling the assumptions above, we see that if* $r+4s+6t>0$ *then*
$$\ell \leq \max\{|s|-1, |t|-1, 11, 2r+8s+6t-1\}$$
and if $r+4s+6t \leq 0$ *then*
$$\ell \leq \max\{|s|-1, |t|-1, 11, 21-8s-12t\}$$

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